

# Kernel estimation of the tail index of a right-truncated Pareto-type distribution

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## Abstract

In this paper, we define a kernel estimator for the tail index of a Pareto-type distribution under random right-truncation and establish its asymptotic normality. A simulation study shows that, compared to the estimators recently proposed by [Gardes and Stupfler \(2015\)](#) and [Benchaira \*et al.\* \(2015b\)](#), this newly introduced estimator behaves better, in terms of bias and mean squared error, for small samples.

**Keywords:** Extreme value index; Heavy-tails; Kernel estimation; Product-limit estimator; Random truncation.

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## 1. Introduction

Let  $(\mathbf{X}_i, \mathbf{Y}_i)$ ,  $1 \leq i \leq N$  be a sample of size  $N \geq 1$  from a couple  $(\mathbf{X}, \mathbf{Y})$  of independent random variables (rv's) defined over some probability space  $(\Omega, \mathcal{A}, \mathbf{P})$ , with continuous marginal distribution functions (df's)  $\mathbf{F}$  and  $\mathbf{G}$  respectively. Suppose that  $\mathbf{X}$  is truncated to the right by  $\mathbf{Y}$ , in the sense that  $\mathbf{X}_i$  is only observed when  $\mathbf{X}_i \leq \mathbf{Y}_i$ . We assume that both survival functions  $\overline{\mathbf{F}} := 1 - \mathbf{F}$  and  $\overline{\mathbf{G}} := 1 - \mathbf{G}$  are regularly varying at infinity with tail indices  $\gamma_1 > 0$  and  $\gamma_2 > 0$  respectively. That is, for any  $x > 0$ ,

$$\lim_{z \rightarrow \infty} \frac{\overline{\mathbf{F}}(xz)}{\overline{\mathbf{F}}(z)} = x^{-1/\gamma_1} \text{ and } \lim_{z \rightarrow \infty} \frac{\overline{\mathbf{G}}(xz)}{\overline{\mathbf{G}}(z)} = x^{-1/\gamma_2}. \quad (1.1)$$

This class of distributions, which includes models such as Pareto, Burr, Fréchet, stable and log-gamma, plays a prominent role in extreme value theory. Also known as heavy-tailed, Pareto-type or Pareto-like distributions, these models have important practical applications and are used rather systematically in certain branches of non-life insurance as well as in finance, telecommunications, geology, and many other fields (see, e.g., [Resnick, 2006](#)). Let us denote  $(X_i, Y_i)$ ,  $i = 1, \dots, n$  to be the observed data, as copies of a couple of rv's  $(X, Y)$ , corresponding to the truncated sample  $(\mathbf{X}_i, \mathbf{Y}_i)$ ,  $i = 1, \dots, N$ , where  $n = n_N$  is a sequence of discrete rv's for which we have, by of the weak law of large numbers

$$n_N/N \xrightarrow{\mathbf{P}} p := \mathbf{P}(\mathbf{X} \leq \mathbf{Y}), \text{ as } N \rightarrow \infty.$$

The joint distribution of  $X_i$  and  $Y_i$  is

$$\begin{aligned} H(x, y) &:= \mathbf{P}(X \leq x, Y \leq y) \\ &= \mathbf{P}(\mathbf{X} \leq \min(x, \mathbf{Y}), \mathbf{Y} \leq y \mid \mathbf{X} \leq \mathbf{Y}) = p^{-1} \int_0^y \mathbf{F}(x, z) d\mathbf{G}(z). \end{aligned}$$

The marginal distributions of the rv's  $X$  and  $Y$ , respectively denoted by  $F$  and  $G$ , are equal to  $F(x) = p^{-1} \int_0^x \overline{\mathbf{G}}(z) d\mathbf{F}(z)$  and  $G(y) = p^{-1} \int_0^y \mathbf{F}(z) d\mathbf{G}(z)$ . The tail of df  $F$  simultaneously depends on  $\overline{\mathbf{G}}$  and  $\overline{\mathbf{F}}$  while that of  $\overline{G}$  only relies on  $\overline{\mathbf{G}}$ . By using Proposition B.1.10 in [de Haan and Ferreira \(2006\)](#), to the regularly varying functions  $\overline{\mathbf{F}}$  and  $\overline{\mathbf{G}}$ , we show that both  $\overline{G}$  and  $\overline{F}$  are regularly varying at infinity as well, with respective indices  $\gamma_2$  and  $\gamma := \gamma_1\gamma_2/(\gamma_1 + \gamma_2)$ . In other words, for any  $s > 0$ ,

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(sx)}{\overline{F}(x)} = s^{-1/\gamma} \text{ and } \lim_{y \rightarrow \infty} \frac{\overline{G}(sy)}{\overline{G}(y)} = s^{-1/\gamma_2}. \quad (1.2)$$

Recently [Gardes and Stupfler \(2015\)](#) addressed the estimation of the extreme value index  $\gamma_1$  under random right-truncation. They used the definition of  $\gamma$  to derive the following

consistent estimator

$$\hat{\gamma}_1^{GS} := k^{-1} \frac{\sum_{i=1}^k \log \frac{X_{n-i+1:n}}{X_{n-k:n}} \sum_{i=1}^k \log \frac{Y_{n-i+1:n}}{Y_{n-k:n}}}{\sum_{i=1}^k \log \frac{X_{n-k:n} Y_{n-i+1:n}}{Y_{n-k:n} X_{n-i+1:n}}}, \quad (1.3)$$

where  $X_{1:n} \leq \dots \leq X_{n:n}$  and  $Y_{1:n} \leq \dots \leq Y_{n:n}$  are the order statistics pertaining to the samples  $(X_1, \dots, X_n)$  and  $(Y_1, \dots, Y_n)$  respectively and  $k = k_n$  is a (random) sequence of discrete rv's satisfying  $k_N \rightarrow \infty$  and  $k_N/N \rightarrow 0$  as  $N \rightarrow \infty$ . The asymptotic normality of  $\hat{\gamma}_1^{GS}$  is established in [Benchaira et al. \(2015a\)](#), under the tail dependence and the second-order regular variation conditions. Also, [Worms and Worms \(2015\)](#) proposed an estimator for  $\gamma_1$  and proved its asymptotic normality, by considering a Lyden-Bell integration with a deterministic threshold. More recently, [Benchaira et al. \(2015b\)](#) treated the case of a random threshold and introduced a Hill-type estimator for the tail index  $\gamma_1$  of randomly right-truncated data, as follows:

$$\hat{\gamma}_1 := \left( \sum_{i=1}^k \frac{\mathbf{F}_n(X_{n-i+1:n})}{C_n(X_{n-i+1:n})} \right)^{-1} \sum_{i=1}^k \frac{\mathbf{F}_n(X_{n-i+1:n})}{C_n(X_{n-i+1:n})} \log \frac{X_{n-i+1:n}}{X_{n-k:n}}, \quad (1.4)$$

where  $\mathbf{F}_n(x) := \prod_{i: X_i > x} \exp \left\{ -\frac{1}{nC_n(X_i)} \right\}$ , is the well-known Woodroffe's product-limit estimator ([Woodroffe, 1985](#)) of the underlying df  $\mathbf{F}$  and

$$C_n(x) := n^{-1} \sum_{i=1}^n \mathbf{1}(X_i \leq x \leq Y_i). \quad (1.5)$$

The asymptotic normality of  $\hat{\gamma}_1$  is established by considering the second-order regular variation conditions (2.13) and (2.14) below and the assumption  $\gamma_1 < \gamma_2$ . The latter condition is required in order to ensure that it remains enough extreme data for the inference to be accurate. In other words, we consider the situation where the tail of the rv of interest  $\mathbf{X}$  is not too contaminated by that of the truncating rv  $\mathbf{Y}$ . Note that, in the presence of complete data, we have  $\mathbf{F}_n \equiv F_n \equiv C_n$  and consequently  $\hat{\gamma}_1$  reduces to the classical Hill estimator ([Hill, 1975](#)). In this paper, we derive a kernel version of  $\hat{\gamma}_1$  in the spirit of what is called kernel estimator of [Csörgő et al. \(1985\)](#). Thereby, for a suitable choice of the kernel function, we obtain an improved estimator of  $\gamma_1$  in terms of bias and mean squared error. To this end, let  $\mathbb{K} : \mathbb{R} \rightarrow \mathbb{R}_+$  be a fixed function, that will be called kernel, satisfying:

- [C1]  $\mathbb{K}$  is non increasing and right-continuous on  $\mathbb{R}$ ;
- [C2]  $\mathbb{K}(s) = 0$  for  $s \notin [0, 1)$  and  $\mathbb{K}(s) \geq 0$  for  $s \in [0, 1)$ ;
- [C3]  $\int_{\mathbb{R}} \mathbb{K}(s) ds = 1$ ;
- [C4]  $\mathbb{K}$  and its first and second Lebesgue derivatives  $\mathbb{K}'$  and  $\mathbb{K}''$  are bounded on  $\mathbb{R}$ .

As examples of such functions (see, e.g., [Groeneboom \*et al.\*, 2003](#)), we have the indicator kernel  $\mathbb{K} = \mathbf{1}_{[0,1]}$  and the biweight and triweight kernels respectively defined by

$$\mathbb{K}_2(s) := \frac{15}{8} (1 - s^2)^2 \mathbf{1}_{\{0 \leq s < 1\}}, \quad \mathbb{K}_3(s) := \frac{35}{16} (1 - s^2)^3 \mathbf{1}_{\{0 \leq s < 1\}}. \quad (1.6)$$

For an overview of kernel estimation of the extreme value index with complete data, one refers to, for instance, [Hüsler \*et al.\* \(2006\)](#) and [Ciuperca and Mercadier \(2010\)](#). By using Potter's inequalities, see e.g. Proposition B.1.10 in [de Haan and Ferreira \(2006\)](#), to the regularly varying function  $\overline{\mathbf{F}}$  together with assumptions [C1]-[C3], we may readily show that

$$\lim_{u \rightarrow \infty} \int_u^\infty x^{-1} \frac{\overline{\mathbf{F}}(x)}{\overline{\mathbf{F}}(u)} \mathbb{K} \left( \frac{\overline{\mathbf{F}}(x)}{\overline{\mathbf{F}}(u)} \right) dx = \gamma_1 \int_0^\infty \mathbb{K}(s) ds = \gamma_1. \quad (1.7)$$

An integration by parts yields

$$\lim_{u \rightarrow \infty} \frac{1}{\overline{\mathbf{F}}(u)} \int_u^\infty g_{\mathbb{K}} \left( \frac{\overline{\mathbf{F}}(x)}{\overline{\mathbf{F}}(u)} \right) \log \frac{x}{u} d\mathbf{F}(x) = \gamma_1, \quad (1.8)$$

where  $g_{\mathbb{K}}$  denotes the Lebesgue derivative of the function  $s \rightarrow \Psi_{\mathbb{K}}(s) := s\mathbb{K}(s)$ . Note that, for  $\mathbb{K} = \mathbf{1}_{[0,1]}$ , we have  $g_{\mathbb{K}} = \mathbf{1}_{[0,1]}$ , then the previous two limits meet assertion (1.2.6) given in Theorem 1.2.2 by [de Haan and Ferreira \(2006\)](#). For kernels  $\mathbb{K}_2$  and  $\mathbb{K}_3$ , we have

$$g_{\mathbb{K}_2}(s) := \frac{15}{8} (1 - s^2) (1 - 5s^2) \mathbf{1}_{\{0 \leq s < 1\}}, \quad g_{\mathbb{K}_3}(s) := \frac{35}{16} (1 - s^2)^2 (1 - 7s^2) \mathbf{1}_{\{0 \leq s < 1\}}. \quad (1.9)$$

Since  $\overline{\mathbf{F}}$  is regularly varying at infinity with tail index  $\gamma > 0$ , then  $X_{n-k:n}$  tends to  $\infty$  almost surely. By replacing, in (1.8),  $u$  by  $X_{n-k:n}$  and  $\mathbf{F}$  by its empirical counterpart  $\mathbf{F}_n$ , we get

$$\widehat{\gamma}_{1,\mathbb{K}} := \frac{1}{\overline{\mathbf{F}}_n(X_{n-k:n})} \int_{X_{n-k:n}}^\infty g_{\mathbb{K}} \left( \frac{\overline{\mathbf{F}}_n(x)}{\overline{\mathbf{F}}_n(X_{n-k:n})} \right) \log \frac{x}{X_{n-k:n}} d\mathbf{F}_n(x),$$

as a kernel estimator for  $\gamma_1$ . Next, we give an explicit formula for  $\widehat{\gamma}_{1,\mathbb{K}}$ . Since  $\overline{\mathbf{F}}$  and  $\overline{\mathbf{G}}$  are regularly varying at infinity with tail indices  $\gamma_1 > 0$  and  $\gamma_2 > 0$  respectively, then their right endpoints are infinite and so they are equal. Hence, from [Woodroffe \(1985\)](#), we may write

$$\int_x^\infty \frac{d\mathbf{F}(y)}{\mathbf{F}(y)} = \int_x^\infty \frac{dF(y)}{C(y)}, \quad (1.10)$$

where  $C(z) := \mathbf{P}(X \leq z \leq Y)$  is the theoretical counterpart of  $C_n(z)$  defined in (1.5). Differentiating (1.10) leads to the following crucial equation  $C(x) d\mathbf{F}(x) = \mathbf{F}(x) dF(x)$ , which implies that

$$C_n(x) d\mathbf{F}_n(x) = \mathbf{F}_n(x) dF_n(x), \quad (1.11)$$

with  $F_n(x) := n^{-1} \sum_{i=1}^n \mathbf{1}(X_i \leq x)$  being the empirical counterpart of  $F(x)$ . This allow us to rewrite  $\hat{\gamma}_{1,\mathbb{K}}$  into

$$\hat{\gamma}_{1,\mathbb{K}} = \frac{1}{\bar{\mathbf{F}}_n(X_{n-k:n})} \int_{X_{n-k:n}}^{\infty} \frac{\mathbf{F}_n(x)}{C_n(x)} g_{\mathbb{K}} \left( \frac{\bar{\mathbf{F}}_n(x)}{\bar{\mathbf{F}}_n(X_{n-k:n})} \right) \log \frac{x}{X_{n-k:n}} dF_n(x),$$

which is equal to

$$\frac{1}{n\bar{\mathbf{F}}_n(X_{n-k:n})} \sum_{i=1}^k \frac{\mathbf{F}_n(X_{n-i+1:n})}{C_n(X_{n-i+1:n})} g_{\mathbb{K}} \left( \frac{\bar{\mathbf{F}}_n(X_{n-i+1:n})}{\bar{\mathbf{F}}_n(X_{n-k:n})} \right) \log \frac{X_{n-i+1:n}}{X_{n-k:n}}.$$

In view of equation (1.11), [Benchaira et al. \(2015b\)](#) showed that

$$\bar{\mathbf{F}}_n(X_{n-k:n}) = \frac{1}{n} \sum_{i=1}^k \frac{\mathbf{F}_n(X_{n-i+1:n})}{C_n(X_{n-i+1:n})}.$$

Thereby, by setting  $a_n^{(i)} := \mathbf{F}_n(X_{n-i+1:n})/C_n(X_{n-i+1:n})$ , we end up with the final formula of our new kernel estimator

$$\hat{\gamma}_{1,\mathbb{K}} = \frac{\sum_{i=1}^k a_n^{(i)} g_{\mathbb{K}} \left( \frac{\bar{\mathbf{F}}_n(X_{n-i+1:n})}{\bar{\mathbf{F}}_n(X_{n-k:n})} \right) \log \frac{X_{n-i+1:n}}{X_{n-k:n}}}{\sum_{i=1}^k a_n^{(i)}}. \quad (1.12)$$

Note that in the complete data situation,  $\mathbf{F}_n$  is equal to  $C_n$  and both reduce to the classical empirical df. As a result, we have in that case  $a_n^{(i)} = 1$  and  $\bar{\mathbf{F}}_n(X_{n-i,n})/\bar{\mathbf{F}}_n(X_{n-k,n}) = i/k$  meaning that  $\hat{\gamma}_{1,\mathbb{K}} = k^{-1} \sum_{i=1}^k g_{\mathbb{K}} \left( \frac{i-1}{k} \right) \log (X_{n-i+1:n}/X_{n-k:n})$ . By applying the mean value theorem to function  $\Psi_{\mathbb{K}}$ , we get

$$\frac{i}{k} \mathbb{K} \left( \frac{i}{k} \right) - \frac{i-1}{k} \mathbb{K} \left( \frac{i-1}{k} \right) = \frac{1}{k} g_{\mathbb{K}} \left( \frac{i-1}{k} \right) + O \left( \frac{1}{k^2} \right), \text{ as } N \rightarrow \infty.$$

It follows that

$$\hat{\gamma}_{1,\mathbb{K}} = \sum_{i=1}^k \left\{ \frac{i}{k} \mathbb{K} \left( \frac{i}{k} \right) - \frac{i-1}{k} \mathbb{K} \left( \frac{i-1}{k} \right) \right\} \log \frac{X_{n-i+1:n}}{X_{n-k:n}} + O \left( \frac{1}{k} \right) \hat{\gamma}_1^{Hill},$$

where  $\hat{\gamma}_1^{Hill} := k^{-1} \sum_{i=1}^k \log(X_{n-i+1:n}/X_{n-k:n})$  is Hill's estimator of the tail index  $\gamma_1$ . In view of the consistency of  $\hat{\gamma}_1^{Hill}$  (Mason, 1982), we obtain

$$\hat{\gamma}_{1,\mathbb{K}} = \sum_{i=1}^k \frac{i}{k} \mathbb{K}\left(\frac{i}{k}\right) \log \frac{X_{n-i+1:n}}{X_{n-i:n}} + O_{\mathbf{P}}\left(\frac{1}{k}\right), \text{ as } N \rightarrow \infty,$$

which is an approximation of the above talked about CDM's kernel estimator of the tail index  $\gamma_1$  with untruncated data. The rest of the paper is organized as follows. In Section 2, we provide our main result, namely the asymptotic normality of  $\hat{\gamma}_{1,\mathbb{K}}$ , whose proof is postponed to Section 4. The finite sample behavior of the proposed estimator is checked by simulation in Section 3, where a comparison with the aforementioned already existing ones is made as well. Finally a lemma that is instrumental to the proof is given in the Appendix.

## 2. Main results

It is very well known that, in the context of extreme value analysis, weak approximations are achieved in the second-order framework (see, e.g., de Haan and Stadtmüller, 1996). Thus, it seems quite natural to suppose that  $\mathbf{F}$  and  $\mathbf{G}$  satisfy the second-order condition of regular variation, which we express in terms of the tail quantile functions pertaining to both df's. That is, we assume that for  $x > 0$ , we have

$$\lim_{t \rightarrow \infty} \frac{\mathbb{U}_{\mathbf{F}}(tx)/\mathbb{U}_{\mathbf{F}}(t) - x^{\gamma_1}}{\mathbf{A}_{\mathbf{F}}(t)} = x^{\gamma_1} \frac{x^{\tau_1} - 1}{\tau_1}, \quad (2.13)$$

and

$$\lim_{t \rightarrow \infty} \frac{\mathbb{U}_{\mathbf{G}}(tx)/\mathbb{U}_{\mathbf{G}}(t) - x^{\gamma_2}}{\mathbf{A}_{\mathbf{G}}(t)} = x^{\gamma_2} \frac{x^{\tau_2} - 1}{\tau_2}, \quad (2.14)$$

where  $\tau_1, \tau_2 < 0$  are the second-order parameters and  $\mathbf{A}_{\mathbf{F}}, \mathbf{A}_{\mathbf{G}}$  are functions tending to zero and not changing signs near infinity with regularly varying absolute values at infinity with indices  $\tau_1, \tau_2$  respectively. For any df  $H$ , the function  $\mathbb{U}_H(t) := H^{\leftarrow}(1 - 1/t)$ ,  $t > 1$ , stands for the tail quantile function, with  $H^{\leftarrow}(u) := \inf\{v : H(v) \geq u\}$ ,  $0 < u < 1$ , denoting the quantile function. For convenience, we set  $\mathbf{A}_{\mathbf{F}}^*(t) := \mathbf{A}_{\mathbf{F}}(1/\overline{\mathbf{F}}(\mathbb{U}_F(t)))$ .

**Theorem 2.1.** *Assume that the second-order conditions of regular variation (2.13) and (2.14) hold with  $\gamma_1 < \gamma_2$ , and let  $\mathbb{K}$  be a kernel function satisfying assumptions [C1]-[C4] and  $k_N$  an integer sequence such that  $k_N \rightarrow \infty$  and  $k_N/N \rightarrow 0$ , as  $N \rightarrow \infty$ . Then, there exist a function  $\mathbf{A}_0(t) \sim \mathbf{A}_{\mathbf{F}}^*(t)$ , as  $t \rightarrow \infty$ , and a standard Wiener process  $\{\mathbf{W}(s); s \geq 0\}$ ,*

defined on the probability space  $(\Omega, \mathcal{A}, \mathbf{P})$  such that

$$\begin{aligned} & \sqrt{k} (\hat{\gamma}_{1,\mathbb{K}} - \gamma_1) \\ &= (\gamma^2/\gamma_1) \int_0^1 s^{-1} \mathbf{W}(s) d\{s\varphi_{\mathbb{K}}(s)\} + \sqrt{k} \mathbf{A}_0(n/k) \int_0^1 s^{-\tau_1} \mathbb{K}(s) ds + o_{\mathbf{P}}(1), \end{aligned}$$

provided that  $\sqrt{k_N} \mathbf{A}_0(N/k_N) = O(1)$ , as  $N \rightarrow \infty$ , where

$$\varphi_{\mathbb{K}}(s) := s^{-1} \int_0^s t^{-\gamma/\gamma_2} \left\{ \mathbb{K}(t^{\gamma/\gamma_1}) - \frac{\gamma_1}{\gamma_2} t^{-\gamma_2/\gamma_1} \mathbb{K}(t^{\gamma/\gamma_1}) + t^{\gamma/\gamma_1} \mathbb{K}'(t^{\gamma/\gamma_1}) \right\} dt.$$

If in addition we suppose that  $\sqrt{k_N} \mathbf{A}_0(N/k_N) \rightarrow \lambda$ , then  $\sqrt{k} (\hat{\gamma}_{1,\mathbb{K}} - \gamma_1) \xrightarrow{\mathcal{D}} \mathcal{N}(\mu_{\mathbb{K}}, \sigma_{\mathbb{K}}^2)$ , as  $N \rightarrow \infty$ , where  $\mu_{\mathbb{K}} := \lambda \int_0^1 s^{-\tau_1} \mathbb{K}(s) ds$  and  $\sigma_{\mathbb{K}}^2 := (\gamma^2/\gamma_1)^2 \int_0^1 \varphi_{\mathbb{K}}^2(s) ds$ .

**Remark 2.1.** A very large value of  $\gamma_2$  yields a  $\gamma$ -value that is very close to  $\gamma_1$ , meaning that the really observed sample is almost the whole dataset. In other words, the complete data case corresponds to the situation when  $1/\gamma_2 \equiv 0$ , in which case we have  $\gamma \equiv \gamma_1$ . It follows that in that case  $\varphi_{\mathbb{K}}(s) = \gamma_1 s^{-1} \int_0^s \{\mathbb{K}(t) + t\mathbb{K}'(t)\} dt = \gamma_1 s^{-1} \int_0^s d\{t\mathbb{K}(t)\} = \gamma_1 \mathbb{K}(s)$ , and therefore  $\sigma_{\mathbb{K}}^2 = \gamma_1^2 \int_0^1 \mathbb{K}^2(s) ds$ , which agrees with the asymptotic variance given in Theorem 1 of [Csörgő et al. \(1985\)](#).

### 3. Simulation study

In this section, we check the finite sample behavior of  $\hat{\gamma}_{1,\mathbb{K}}$  and, at the same time, we compare it with  $\hat{\gamma}_1$  and  $\hat{\gamma}_1^{(\text{GS})}$  respectively proposed by [Benchaira et al. \(2015b\)](#) and [Gardes and Stupfler \(2015\)](#) and defined in (1.4) and (1.3). To this end, we consider two sets of truncated and truncation data, both drawn from Burr's model:

$$\overline{\mathbf{F}}(x) = (1 + x^{1/\delta})^{-\delta/\gamma_1}, \quad \overline{\mathbf{G}}(x) = (1 + x^{1/\delta})^{-\delta/\gamma_2}, \quad x \geq 0,$$

where  $\delta, \gamma_1, \gamma_2 > 0$ . The corresponding percentage of observed data is equal to  $p = \gamma_2/(\gamma_1 + \gamma_2)$ . We fix  $\delta = 1/4$  and choose the values 0.6 and 0.8 for  $\gamma_1$  and 70%, 80% and 90% for  $p$ . For each couple  $(\gamma_1, p)$ , we solve the equation  $p = \gamma_2/(\gamma_1 + \gamma_2)$  to get the pertaining  $\gamma_2$ -value. For the construction of our estimator  $\hat{\gamma}_{1,\mathbb{K}}$ , we select the biweight and the triweight kernel functions defined in (1.6). We vary the common size  $N$  of both samples  $(\mathbf{X}_1, \dots, \mathbf{X}_N)$  and  $(\mathbf{Y}_1, \dots, \mathbf{Y}_N)$ , then for each size, we generate 1000 independent replicates. Our overall results are taken as the empirical means of the results obtained through all repetitions. To determine the optimal number of top statistics used in the computation of each one of the three estimators, we use the algorithm of [Reiss and Thomas \(2007\)](#), page 137. Our illustration and comparison are made with respect to the estimators absolute biases (abs

bias) and the roots of their mean squared errors (rmse). We summarize the simulation results in Tables 3.1 and 3.2 for  $\gamma_1 = 0.6$  and in Tables 3.3 and 3.4 for  $\gamma_1 = 0.8$ . In light of all four tables, we first note that, as expected, the estimation accuracy of all estimators decreases when the truncation percentage increases. Second, with regard to the bias, the comparison definitely is in favour of the newly proposed tail index estimator  $\hat{\gamma}_{1,\mathbb{K}}$ , whereas it is not as clear-cut when the rmse is considered. Indeed, the kernel estimator preforms better than the other pair as far as small samples are concerned while for large datasets, it is  $\hat{\gamma}_1^{(\text{GS})}$  that seems to have the least rmse but with greater bias. As an overall conclusion, one may say that, for case studies where not so many data are at one's disposal, the kernel estimator  $\hat{\gamma}_{1,\mathbb{K}}$  is the most suitable among the three estimators.

#### 4. Proofs

The proof is based on a useful weak approximation to the tail product-limit process recently provided by [Benchaira et al. \(2015b\)](#). From (1.7), the estimator  $\hat{\gamma}_{1,\mathbb{K}}$  may be rewritten into

$$\hat{\gamma}_{1,\mathbb{K}} = \int_1^\infty x^{-1} \Psi_{\mathbb{K}} \left( \frac{\bar{\mathbf{F}}_n(xX_{n-k:n})}{\bar{\mathbf{F}}_n(X_{n-k:n})} \right) dx.$$

Recall that  $\Psi_{\mathbb{K}}(s) = s\mathbb{K}(s)$ , then it is easy to verify that  $\int_1^\infty x^{-1} \Psi_{\mathbb{K}}(x^{-1/\gamma_1}) dx = \gamma_1$ . Hence

$$\hat{\gamma}_{1,\mathbb{K}} - \gamma_1 = \int_1^\infty x^{-1} \left\{ \Psi_{\mathbb{K}} \left( \frac{\bar{\mathbf{F}}_n(xX_{n-k:n})}{\bar{\mathbf{F}}_n(X_{n-k:n})} \right) - \Psi_{\mathbb{K}}(x^{-1/\gamma_1}) \right\} dx.$$

Let

$$\mathbf{D}_n(x) := \sqrt{k} \left( \frac{\bar{\mathbf{F}}_n(xX_{n-k:n})}{\bar{\mathbf{F}}_n(X_{n-k:n})} - x^{-1/\gamma_1} \right), \quad x > 0, \quad (4.15)$$

be the tail product-limit process, then Taylor's expansion of  $\Psi_{\mathbb{K}}$  yields that

$$\sqrt{k} (\hat{\gamma}_{1,\mathbb{K}} - \gamma_1) = \int_1^\infty x^{-1} \mathbf{D}_n(x) g_{\mathbb{K}}(x^{-1/\gamma_1}) dx + R_{n1},$$

with  $R_{n1} := 2^{-1}k^{-1/2} \int_1^\infty x^{-1} \mathbf{D}_n^2(x) g'_{\mathbb{K}}(\xi_n(x)) dx$ , where  $\xi_n(x)$  is a stochastic intermediate value lying between  $\bar{\mathbf{F}}_n(xX_{n-k:n})/\bar{\mathbf{F}}_n(X_{n-k:n})$  and  $x^{-1/\gamma_1}$ . According to [Benchaira et al. \(2015b\)](#), we have, for  $0 < \epsilon < 1/2 - \gamma/\gamma_2$

$$\sup_{x \geq 1} x^{(1/2-\epsilon)/\gamma-1/\gamma_2} \left| \mathbf{D}_n(x) - \boldsymbol{\Gamma}(x; \mathbf{W}) - x^{-1/\gamma_1} \frac{x^{\tau_1/\gamma_1} - 1}{\gamma_1 \tau_1} \sqrt{k} \mathbf{A}_0(n/k) \right| \xrightarrow{\mathbf{P}} 0, \quad \text{as } N \rightarrow \infty, \quad (4.16)$$



$p = 0.7$							
		$\hat{\gamma}_{1,\mathbb{K}}$		$\hat{\gamma}_1$		$\hat{\gamma}_1^{GS}$	
$N$	$n$	abs bias	rmse	abs bias	rmse	abs bias	rmse
150	104	0.073	0.665	0.133	0.408	0.136	3.341
200	140	0.008	0.614	0.152	0.392	0.258	1.647
300	210	0.003	0.467	0.095	0.321	0.102	0.962
500	349	0.007	0.439	0.063	0.296	0.022	0.409
1000	699	0.020	0.284	0.042	0.210	0.023	0.211
1500	1049	0.009	0.255	0.024	0.189	0.013	0.142
2000	1399	0.011	0.245	0.018	0.177	0.013	0.116

  

$p = 0.8$							
150	120	0.054	0.608	0.093	0.398	0.100	0.989
200	160	0.030	0.520	0.085	0.353	0.109	0.488
300	239	0.022	0.467	0.067	0.322	0.069	0.353
500	399	0.002	0.340	0.049	0.240	0.040	0.196
1000	799	0.013	0.217	0.033	0.168	0.029	0.135
1500	1199	0.003	0.190	0.017	0.140	0.019	0.109
2000	1599	0.005	0.149	0.011	0.113	0.005	0.095

  

$p = 0.9$							
150	134	0.031	0.492	0.082	0.387	0.149	2.740
200	180	0.019	0.404	0.069	0.313	0.072	0.334
300	270	0.016	0.299	0.051	0.238	0.043	0.231
500	449	0.002	0.236	0.045	0.176	0.037	0.160
1000	899	0.006	0.163	0.024	0.131	0.020	0.123
1500	1350	0.010	0.131	0.021	0.103	0.018	0.093
2000	1799	0.002	0.116	0.010	0.088	0.009	0.078

TABLE 3.1. Biweight-kernel estimation results for the shape parameter  $\gamma_1 = 0.6$  of Burr's model based on 1000 right-truncated samples, along with other existing estimators

$p = 0.7$							
		$\hat{\gamma}_{1,\mathbb{K}}$		$\hat{\gamma}_1$		$\hat{\gamma}_1^{GS}$	
$N$	$n$	abs bias	rmse	abs bias	rmse	abs bias	rmse
150	104	0.134	0.808	0.142	0.408	0.245	1.242
200	139	0.097	0.705	0.129	0.373	0.184	0.857
300	209	0.045	0.566	0.090	0.313	0.091	0.582
500	349	0.002	0.430	0.074	0.268	0.064	0.550
1000	699	0.003	0.399	0.031	0.237	0.023	0.161
1500	1050	0.010	0.362	0.013	0.217	0.010	0.130
2000	1401	0.010	0.244	0.018	0.164	0.009	0.117

  

$p = 0.8$							
150	119	0.096	0.730	0.109	0.397	0.117	0.729
200	159	0.060	0.580	0.091	0.340	0.108	0.874
300	239	0.037	0.496	0.067	0.315	0.080	0.490
500	399	0.009	0.303	0.057	0.231	0.047	0.280
1000	799	0.001	0.265	0.027	0.177	0.021	0.139
1500	1199	0.008	0.194	0.018	0.139	0.015	0.109
2000	1600	0.001	0.183	0.013	0.124	0.012	0.095

  

$p = 0.9$							
150	134	0.066	0.660	0.080	0.392	0.081	0.450
200	179	0.047	0.454	0.061	0.314	0.061	0.359
300	270	0.003	0.299	0.064	0.243	0.062	0.230
500	449	0.001	0.226	0.043	0.174	0.037	0.164
1000	899	0.009	0.175	0.016	0.124	0.014	0.113
1500	1350	0.002	0.146	0.017	0.108	0.017	0.098
2000	1799	0.003	0.134	0.010	0.093	0.008	0.081

TABLE 3.2. Triweight-kernel estimation results for the shape parameter  $\gamma_1 = 0.6$  of Burr's model based on 1000 right-truncated samples, along with other existing estimators

$p = 0.7$							
		$\hat{\gamma}_{1,\mathbb{K}}$		$\hat{\gamma}_1$		$\hat{\gamma}_1^{GS}$	
$N$	$n$	abs bias	rmse	abs bias	rmse	abs bias	rmse
150	105	0.090	0.893	0.187	0.548	0.294	2.126
200	139	0.014	0.863	0.199	0.542	0.316	1.351
300	210	0.022	0.573	0.140	0.412	0.173	0.812
500	349	0.031	0.519	0.103	0.372	0.053	0.593
1000	699	0.004	0.462	0.042	0.324	0.020	0.253
1500	1049	0.017	0.356	0.031	0.255	0.020	0.174
2000	1399	0.008	0.424	0.017	0.267	0.017	0.150

  

$p = 0.8$							
150	120	0.088	0.862	0.122	0.553	0.248	1.947
200	159	0.040	0.684	0.121	0.472	0.178	1.143
300	239	0.006	0.516	0.084	0.406	0.099	0.494
500	399	0.022	0.372	0.078	0.285	0.058	0.247
1000	800	0.003	0.297	0.029	0.221	0.021	0.189
1500	1199	0.004	0.239	0.020	0.180	0.012	0.157
2000	1599	0.001	0.209	0.013	0.156	0.014	0.121

  

$p = 0.9$							
150	134	0.034	0.585	0.113	0.479	0.118	0.543
200	180	0.002	0.512	0.120	0.402	0.127	0.459
300	270	0.003	0.389	0.082	0.320	0.073	0.310
500	450	0.002	0.305	0.052	0.246	0.045	0.228
1000	900	0.004	0.223	0.024	0.169	0.020	0.153
1500	1349	0.005	0.176	0.020	0.141	0.021	0.124
2000	1800	0.006	0.166	0.013	0.126	0.013	0.110

TABLE 3.3. Biweight-kernel estimation results for the shape parameter  $\gamma_1 = 0.8$  of Burr's model based on 1000 right-truncated samples, along with other existing estimators

$p = 0.7$							
		$\hat{\gamma}_{1,\mathbb{K}}$		$\hat{\gamma}_1$		$\hat{\gamma}_1^{GS}$	
$N$	$n$	abs bias	rmse	abs bias	rmse	abs bias	rmse
150	104	0.159	0.976	0.202	0.511	0.386	3.264
200	139	0.064	0.905	0.205	0.493	0.247	1.355
300	209	0.090	0.831	0.101	0.469	0.141	1.082
500	349	0.014	0.589	0.090	0.371	0.063	0.586
1000	700	0.013	0.458	0.049	0.296	0.023	0.264
1500	1050	0.008	0.561	0.023	0.315	0.020	0.189
2000	1400	0.012	0.381	0.027	0.241	0.013	0.164

  

$p = 0.8$							
150	120	0.103	0.886	0.151	0.511	0.180	1.906
200	160	0.058	0.775	0.131	0.466	0.153	1.311
300	239	0.023	0.629	0.106	0.398	0.078	0.502
500	399	0.005	0.515	0.069	0.339	0.060	0.256
1000	800	0.005	0.330	0.036	0.226	0.030	0.186
1500	1200	0.017	0.242	0.035	0.176	0.029	0.145
2000	1600	0.001	0.225	0.017	0.160	0.012	0.133

  

$p = 0.9$							
150	135	0.039	0.611	0.117	0.465	0.133	1.103
200	180	0.047	0.603	0.102	0.435	0.127	0.845
300	270	0.020	0.414	0.078	0.308	0.071	0.301
500	449	0.008	0.321	0.049	0.256	0.050	0.223
1000	900	0.011	0.230	0.024	0.173	0.020	0.153
1500	1350	0.008	0.197	0.016	0.137	0.015	0.120
2000	1800	0.001	0.162	0.014	0.115	0.011	0.105

TABLE 3.4. Triweight-kernel estimation results for the shape parameter  $\gamma_1 = 0.8$  of Burr's model based on 1000 right-truncated samples, along with other existing estimators

where  $\{\Gamma(x; \mathbf{W}); x > 0\}$  is a Gaussian process defined by

$$\begin{aligned}\Gamma(x; \mathbf{W}) &:= \frac{\gamma}{\gamma_1} x^{-1/\gamma_1} \{x^{1/\gamma} \mathbf{W}(x^{-1/\gamma}) - \mathbf{W}(1)\} \\ &\quad + \frac{\gamma}{\gamma_1 + \gamma_2} x^{-1/\gamma_1} \int_0^1 s^{-\gamma/\gamma_2 - 1} \{x^{1/\gamma} \mathbf{W}(x^{-1/\gamma} s) - \mathbf{W}(s)\} ds.\end{aligned}$$

Now, we write  $\sqrt{k}(\hat{\gamma}_{1,K} - \gamma_1) = \int_1^\infty x^{-1} \Gamma(x; \mathbf{W}) g_{\mathbb{K}}(x^{-1/\gamma_1}) dx + \sum_{i=1}^3 R_{ni}$ , where

$$R_{n2} := \int_1^\infty x^{-1} \left\{ \mathbf{D}_n(x) - \Gamma(x; \mathbf{W}) - x^{-1/\gamma_1} \frac{x^{\tau_1/\gamma_1} - 1}{\gamma_1 \tau_1} \sqrt{k} \mathbf{A}_0(n/k) \right\} g_{\mathbb{K}}(x^{-1/\gamma_1}) dx,$$

and

$$R_{n3} := \int_1^\infty x^{-1} \left\{ x^{-1/\gamma_1} \frac{x^{\tau_1/\gamma_1} - 1}{\gamma_1 \tau_1} \sqrt{k} \mathbf{A}_0(n/k) \right\} g_{\mathbb{K}}(x^{-1/\gamma_1}) dx.$$

Elementary calculation yields that

$$\int_1^\infty x^{-1} \Gamma(x; \mathbf{W}) g_{\mathbb{K}}(x^{-1/\gamma_1}) dx = (\gamma^2/\gamma_1) \int_0^1 s^{-1} \mathbf{W}(s) d\{s\varphi_{\mathbb{K}}(s)\} =: Z,$$

where  $\varphi_{\mathbb{K}}(s)$  is that defined in the theorem. Next, we evaluate the remainder terms  $R_{ni}$ ,  $i = 1, 2, 3$ . First, we show that  $R_{n1}$  tends to zero in probability, as  $N \rightarrow \infty$ . Recall that  $\gamma_1 < \gamma_2$  and  $0 < \epsilon < 1/2 - \gamma/\gamma_2$ , then  $(1/2 - \epsilon)/\gamma - 1/\gamma_2 > 0$ . It follows that  $\int_1^\infty x^{2(1/\gamma_2 - (1/2 - \epsilon)/\gamma) - 1} dx$  is finite and, from Lemma 5.1, we get  $\sup_{x \geq 1} |\mathbf{D}_n^2(x)| = O_{\mathbf{P}}(1)$ . On the other hand, from assumption  $[\mathbb{C}4]$ , we infer that  $g'_{\mathbb{K}}$  is bounded on  $(0, 1)$ . Consequently, we have  $R_{n1} = o_{\mathbf{P}}(1)$ . Second, for the term  $R_{n2}$ , we use approximation (4.16), to get

$$R_{n2} = o_{\mathbf{P}}(1) \int_1^\infty x^{1/\gamma_2 - (1/2 - \epsilon)/\gamma - 1} |g_{\mathbb{K}}(x^{-1/\gamma_1})| dx.$$

Since  $g_{\mathbb{K}}$  is bounded on  $(0, 1)$ , then  $R_{n2} = o_{\mathbf{P}}(1)$ . Finally, we show that the third term  $R_{n3}$  is equal to  $\sqrt{k} \mathbf{A}_0(n/k) \int_0^1 s^{-\tau_1} \mathbb{K}(s) ds$ . Observe that

$$R_{n3} = \sqrt{k} \mathbf{A}_0(n/k) \int_1^\infty x^{-1/\gamma_1 - 1} \frac{x^{\tau_1/\gamma_1} - 1}{\gamma_1 \tau_1} g_{\mathbb{K}}(x^{-1/\gamma_1}) dx.$$

By using a change of variables and by replacing  $\Psi_{\mathbb{K}}(s) = s\mathbb{K}(s)$ , we end up with

$$\int_1^\infty x^{-1/\gamma_1 - 1} \frac{x^{\tau_1/\gamma_1} - 1}{\gamma_1 \tau_1} g_{\mathbb{K}}(x^{-1/\gamma_1}) dx = \int_0^1 s^{-\tau_1} \mathbb{K}(s) ds.$$

For the second part of the theorem, it suffices to use Lemma 8 in Csörgő *et al.* (1985), to show that the variance of the centred Gaussian rv  $Z$  equals  $\sigma_{\mathbb{K}}^2$ . Finally, whenever  $\sqrt{k_N} \mathbf{A}_0(N/k_N) \rightarrow \lambda$ , we have  $R_{n3} \xrightarrow{\mathbf{P}} \lambda \int_0^1 s^{-\tau_1} \mathbb{K}(s) ds$ , as  $N \rightarrow \infty$ , which corresponds to the asymptotic bias  $\mu_{\mathbb{K}}$  as sought.

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## 5. APPENDIX

**Lemma 5.1.** *Under the assumptions of Theorem 2.1, we have, for any  $0 < \epsilon < 1/2 - \gamma/\gamma_2$*

$$\sup_{x \geq 1} x^{(1/2-\epsilon)/\gamma-1/\gamma_2} |\mathbf{D}_n(x)| = O_{\mathbf{p}}(1), \text{ as } N \rightarrow \infty.$$

*Proof.* This result is straightforward from the weak approximation (4.16). Indeed, it is clear that  $\sup_{x \geq 1} x^{(1/2-\epsilon)/\gamma-1/\gamma_2} |\mathbf{D}_n(x)| \leq T_{1,n} + T_{2,n} + T_3$ , where

$$T_{1,n} := \sup_{x \geq 1} x^{(1/2-\epsilon)/\gamma-1/\gamma_2} \left| \mathbf{D}_n(x) - \Gamma(x; \mathbf{W}) - x^{-1/\gamma_1} \frac{x^{\tau_1/\gamma_1} - 1}{\gamma_1 \tau_1} \sqrt{k} \mathbf{A}_0(n/k) \right|,$$

$$T_{2,n} := \frac{\sqrt{k} \mathbf{A}_0(n/k)}{\gamma_1 \tau_1} \sup_{x \geq 1} \left\{ x^{-(1/2+\epsilon)/\gamma} (1 - x^{\tau_1/\gamma_1}) \right\} \text{ and } T_3 := \sup_{x \geq 1} x^{(1/2-\epsilon)/\gamma-1/\gamma_2} |\Gamma(x; \mathbf{W})|.$$

First, it is readily checked from (4.16) that  $T_{1,n} = o_{\mathbf{p}}(1)$ . Second, observe that, in addition to the assumption  $\sqrt{k} \mathbf{A}_0(n/k) = O_{\mathbf{p}}(1)$ , we have  $x^{-(1/2+\epsilon)/\gamma} (1 - x^{\tau_1/\gamma_1}) \leq 2$ , for  $x \geq 1$ , it follows that  $T_{2,n} = O_{\mathbf{p}}(1)$ . Finally, note that  $x^{(1/2-\epsilon)/\gamma-1/\gamma_2} \Gamma(x; \mathbf{W})$  is equal to

$$\begin{aligned} & x^{-(1/2+\epsilon)/\gamma} \left\{ \frac{\gamma}{\gamma_1} (x^{1/\gamma} \mathbf{W}(x^{-1/\gamma}) - \mathbf{W}(1)) \right. \\ & \quad \left. + \frac{\gamma}{\gamma_1 + \gamma_2} \int_0^1 s^{-\gamma/\gamma_2-1} (x^{1/\gamma} \mathbf{W}(x^{-1/\gamma} s) - \mathbf{W}(s)) ds \right\}, \end{aligned}$$

where the quantity between brackets is a Gaussian rv and  $x^{-(1/2+\epsilon)/\gamma} \leq 1$ , for  $x \geq 1$ . Therefore,  $T_3 = O_{\mathbf{p}}(1)$  and the proof is completed.  $\square$